DOI 10.1007/s11253-017-1291-0

*Ukrainian Mathematical Journal, Vol.68, No.8, January, 2017 (Ukrainian Original Vol. 68, No. 8, August, 2016)*

# ASYMPTOTICS OF NORMALIZED CONTROL WITH MARKOV SWITCHINGS

**A. V. Nikitin**[1] **and U. T. Khimka**[2]

UDC 519.21

We study the process of transfer of Markov perturbations and control over this process under the condition of existence of the equilibrium point of the quality criterion. For this control, we construct a normalized process and establish its asymptotic normality in the form of Ornstein–Uhlenbeck process in the case where the transfer process varies under the influence of Markov switchings along a new trajectory of evolution from the state in which it was at the time of switching.

## Introduction

For the transfer processes described by the stochastic differential equation [1] with diffusion control process, the conditions of existence of this control were obtained in [2]. A special case of existence of an equilibrium point of the quality criterion is realized in numerous applied problems [3, 4]. A procedure of stochastic approximation for this control specifying the conditions of convergence to the equilibrium point of the quality criterion can also be considered for this control [3]. In this case, an independent problem is connected with the determination of the law of distribution of the limiting normalized control process under the conditions of convergence of the constructed procedure [4, 5]. Thus, new results of application of a small parameter and the solution of the problem of singular perturbation [6] enable one to establish the asymptotic normality of the procedure of stochastic approximation with Markov perturbations [7] and semi-Markov switchings [8] for the corresponding normalization both in time and in a small parameter $\varepsilon > 0$.

In the present paper, we consider a transfer process with Markov perturbations and control over the conditions of existence of an equilibrium point of the quality criterion with Markov switchings [6]. For this control, we construct a normalized process and establish its asymptotic normality in the form of an Ornstein–Uhlenbeck process.

We consider the case where the transfer process, i.e., a random evolution, varies under the influence of Markov switchings along the trajectory of new evolution from the state in which it was at the time of switching (regarded as the initial state) [6].

## Statement of the Problem

Assume that the transfer process $y^{\varepsilon}(t) \in \mathrm{R}^d$ is determined by the following stochastic differential equation:

$$dy^{\varepsilon}(t) = a\left( y^{\varepsilon}(t), x\left( \frac{t}{\varepsilon^2} \right) \right) dt + \sigma\left( y^{\varepsilon}(t), x\left( \frac{t}{\varepsilon^2} \right), u^{\varepsilon}(t) \right) dw(t), \tag{1}$$

[1] Shevchenko Kiev National University, Kiev, Ukraine.

[2] “L'vivs'ka Politekhnika” National University, Lviv, Ukraine.

Translated from Ukrains’kyi Matematychnyi Zhurnal, Vol. 68, No. 8, pp. 1092–1101, August, 2016. Original article submitted October 24, 2015; revision submitted June 1, 2016.

 0041-5995/17/6808–1252 

where $x(t)$, $t > 0$, is a uniformly ergodic Markov process in a measurable phase space of states $(X, \boldsymbol{X})$ specified by the generator [6]

$$\boldsymbol{Q}\varphi(x) = q(x)\int_X P(x,dy)[\varphi(y) - \varphi(x)]$$

in a Banach space $B(X)$ of real-valued bounded functions $\varphi(x)$ with supremum-norm

$$\|\varphi(x)\| = \sup_{x \in X} |\varphi(x)|.$$

The generator $\boldsymbol{Q}$ is reducibly invertible on $B(X)$ with the projector

$$\Pi\varphi(x) := \int_X \pi(dx)\varphi(x),$$

where $\pi(B)$, $B \in \boldsymbol{X}$, is a stationary distribution of the Markov process $x(t)$, $t \geq 0$, given by the formula

$$\pi(dx)q(x) = qp(dx), \qquad q = \int_X \pi(dx)q(x)$$

($\rho(dx)$ is a stationary distribution of the embedded Markov chain $x_n$, $n \geq 0$) and by the potential $\mathbf{R}_0$ with the following operator representation:

$$\mathbf{R}_0 = \Pi - [\mathbf{Q} + \Pi]^{-1}.$$

The functions

$$a(y,x) = (a_k(y,x), k = \overline{1,d}) \quad \text{and} \quad \sigma(y,x,u) = (\sigma_k(y,x,u), k = \overline{1,d}), \qquad y \in \mathrm{R}^d, \quad x \in X,$$

satisfy the conditions of existence of a global solution of the evolutionary equations

$$dy_x(t) = a(y_x(t),x)dt + \sigma(y_x(t),x,u_x(t))dw(t), \quad x \in X,$$

for every fixed value of $x$ of the Markov process $x(t)$, $t \geq 0$, on the segment $[\tau_i, \tau_{i+1}]$ of stay of the process $x(t)$, $t \geq 0$, in the state $x \in X$.

Assume that the control $u(t)$ in the general representation (1) is specified by the condition

$$du^{\varepsilon}(t) = \alpha(t)G\left(y^{\varepsilon}(t), x\left(\frac{t}{\varepsilon^2}\right), u^{\varepsilon}(t)\right)dt, \tag{2}$$

where the conditions imposed on the function $\alpha(t)$ take the form

$$\int_0^\infty \alpha(t)\,dt = \infty \quad \text{and} \quad \int_0^\infty \alpha^2(t)\,dt < \infty. \tag{3}$$

In particular, conditions (3) are satisfied for

$$\alpha(t) = \frac{\alpha}{t}$$

(this is considered in what follows).

Note [3, 7, 8] that conditions (3) guarantee convergence of the control $u^{\varepsilon}(t)$ to the equilibrium point of the quality criterion, i.e., to the point $u^*$ determined from the condition

$$G(y, u^*) = 0,$$

$$G(y, u) = \int_X \pi(dx) G(y, x, u).$$

The normalized control has the form

$$v^{\varepsilon}(t) = \frac{\sqrt{t}}{\varepsilon} u^{\varepsilon}(t) \tag{4}$$

and the balance condition takes the form

$$\Pi G(y, x, 0) = \int_X \pi(dx) G(y, x, 0) = 0. \tag{5}$$

**Theorem 1.** *Under the conditions of convergence (3) for problem (1), (2) and the additional conditions*

($D_1$) $\hat{\sigma}_v^2(y) = 2\int_X \pi(dx)\, G(y, x, 0)\, \mathbf{R}_0 G(y, x, 0) > 0,$

($D_2$) $\alpha g(y) < -\frac{1}{2}, \quad g(y) = \int_X \pi(dx)\, G'_V(y, x, 0),$

*the weak convergence*

$$v^{\varepsilon}(t) \Rightarrow \zeta(t), \qquad \varepsilon \to 0,$$

*is realized in every finite interval* $(0 < t_0 < t < T)$.

*The limiting process* $\zeta(t)$, $t > 0$, *is the Ornstein–Uhlenbeck process specified by the generator*

$$\mathbf{L}_v \varphi(y, v) = v\left(\alpha g(y) + \frac{1}{2}\right)\varphi'_v(y, v) + \frac{1}{2}\alpha^2 \hat{\sigma}_v^2(y)\, \varphi''_{vv}(y, v).$$

We now establish several auxiliary properties of the transfer process $y^{\varepsilon}(t)$ and the normalized control $v^{\varepsilon}(t)$.

**Lemma 1.** *The processes* $y^{\varepsilon}(t)$ *and* $v^{\varepsilon}(t)$ *are solutions of the stochastic differential equations*

$$dy^{\varepsilon}(t) = a(y^{\varepsilon}(t), x_t^{\varepsilon})dt + \sigma\left( y^{\varepsilon}(t), x_t^{\varepsilon}, \frac{\varepsilon}{\sqrt{t}} v^{\varepsilon}(t) \right) dw(t), \tag{6}$$

$$dv^{\varepsilon}(t) = \varepsilon^{-1} \frac{\alpha}{\sqrt{t}} G\left( y^{\varepsilon}(t), x_t^{\varepsilon}, \frac{\varepsilon}{\sqrt{t}} v^{\varepsilon}(t) \right) dt + \frac{v^{\varepsilon}(t)}{2t} dt, \tag{7}$$

*where*

$$x_t^{\varepsilon} := x\left( \frac{t}{\varepsilon^2} \right).$$

***Proof.*** By using relation (4), we get

$$u^{\varepsilon}(t) = \frac{\varepsilon}{\sqrt{t}} v^{\varepsilon}(t).$$

Thus, in view of (2), we obtain (5) and (6).

**Lemma 2.** *The generator of a three-component Markov process*

$$y_t^{\varepsilon} := y^{\varepsilon}(t), \quad x_t^{\varepsilon} := x\left( \frac{t}{\varepsilon^2} \right), \quad u_t^{\varepsilon} := u^{\varepsilon}(t), \quad t \ge t_0 > 0, \tag{8}$$

*has the* form

$$\boldsymbol{L}_t^{\varepsilon}(x)\varphi(y,x,v) = \varepsilon^{-2}\boldsymbol{Q}\varphi(y,x,v) + \boldsymbol{V}_t^{\varepsilon}(x)\varphi(y,x,v), \tag{9}$$

*where*

$$\boldsymbol{V}_t^{\varepsilon}(x)\,\varphi(y,x,v) = \left[ \varepsilon^{-1} \frac{\alpha}{\sqrt{t}} G\left( y, x, \frac{\varepsilon}{\sqrt{t}} v \right) + \frac{v}{2t} \right] \varphi_v'(y,x,v)$$

$$+ \; \varphi_y'(y,x,v)\, a(y,x) + \frac{1}{2} \varphi_{yy}''(y,x,v)\, \sigma^2\left( y, x, \frac{\varepsilon}{\sqrt{t}} v \right).$$

***Proof.*** To construct the generator of process (8), we determine the conditional expectation

$$E\left[ \varphi(y_{t+\Delta}^{\varepsilon}, x_{t+\Delta}^{\varepsilon}, v_{t+\Delta}^{\varepsilon}) - \varphi(y,x,v) \Big|_{y^{\varepsilon}(t)=y,\; x_t^{\varepsilon}=x,\; v^{\varepsilon}(t)=v} \right]$$

$$= E_{y,x,v}\left[ \varphi(y + \Delta y^{\varepsilon}, x_{t+\Delta}^{\varepsilon}, v + \Delta v^{\varepsilon}) - \varphi(y,x,v) \right]$$

$$= E_{y,x,v}\Big[\varphi(y+\Delta y^{\varepsilon}, x, v+\Delta v^{\varepsilon}) - \varphi(y,x,v)\Big] I(\theta > \varepsilon^{-2}\Delta)$$

$$+ E_{y,x,v}\Big[\varphi(y+\Delta y^{\varepsilon}, x^{\varepsilon}_{t+\Delta}, v+\Delta v^{\varepsilon}) - \varphi(y,x,v)\Big] I(\theta < \varepsilon^{-2}\Delta), \qquad (10)$$

where $\theta$ is the time of stay of the Markov process $x(t)$, $t > 0$, in the state $x$.

Further, we take into account the representations

$$I(\theta \ge \varepsilon^{-2}\Delta) = 1 - \varepsilon^{-2} q(x)\Delta + o(\Delta),$$

$$I(\theta < \varepsilon^{-2}\Delta) = \varepsilon^{-2} q(x)\Delta + o(\Delta).$$

It follows from Eq. (6) that

$$y^{\varepsilon}_{t+\Delta} = y + \Delta y^{\varepsilon} = y + \int_t^{t+\Delta} a\big(y^{\varepsilon}(s), x^{\varepsilon}_s\big)\, ds$$

$$+ \int_t^{t+\Delta} \sigma\left( y^{\varepsilon}(s), x^{\varepsilon}_s, \frac{\varepsilon}{\sqrt{s}} v^{\varepsilon}(s) \right) dw(s) = \bar{y} + \int_t^{t+\Delta} \sigma\left( y^{\varepsilon}(s), x^{\varepsilon}_s, \frac{\varepsilon}{\sqrt{s}} v^{\varepsilon}(s) \right) dw(s),$$

where

$$\bar{y} = y + \int_t^{t+\Delta} a\big(y^{\varepsilon}(s), x^{\varepsilon}_s\big)\, ds.$$

Consider

$$\varphi\big(y+\Delta y^{\varepsilon}, x, v+\Delta v^{\varepsilon}\big) = \varphi\left( \bar{y} + \int_t^{t+\Delta} \sigma\left( y^{\varepsilon}(s), x^{\varepsilon}_s, \frac{\varepsilon}{\sqrt{s}} v^{\varepsilon}(s) \right) dw(s), x, v+\Delta v^{\varepsilon} \right)$$

$$= \varphi\,(\bar{y}, x, v+\Delta v^{\varepsilon}) + \varphi'_y(\bar{y}, x, v+\Delta v^{\varepsilon}) \int_t^{t+\Delta} \sigma\left( y^{\varepsilon}(s), x^{\varepsilon}_s, \frac{\varepsilon}{\sqrt{s}} v^{\varepsilon}(s) \right) dw(s)$$

$$+ \frac{1}{2}\varphi''_{yy}(\bar{y}, x, v+\Delta v^{\varepsilon}) \left[ \int_t^{t+\Delta} \sigma\left( y^{\varepsilon}(s), x^{\varepsilon}_s, \frac{\varepsilon}{\sqrt{s}} v^{\varepsilon}(s) \right) dw(s) \right]^2 + o(\Delta).$$

By virtue of (7), we find

$$\varphi'_y(\bar{y}, x, v+\Delta v^{\varepsilon}) = \varphi'_y(\bar{y}, x, v) + \varphi''_{yv}(\bar{y}, x, v)\left[ \varepsilon^{-1}\frac{\alpha}{\sqrt{t}} G(y,x,v) + \frac{v}{t} \right]\Delta + o(\Delta)$$

and

$$\varphi(\bar{y}, x, v + \Delta v^{\varepsilon}) = \varphi(y, x, v + \Delta v^{\varepsilon})$$

$$+ \varphi'_y(y, x, v + \Delta v^{\varepsilon})\, a(y, x)\Delta + o(\Delta)$$

$$= \varphi(y, x, v) + \varphi'_v(y, x, v)\left[\varepsilon^{-1}\frac{\alpha}{\sqrt{t}} G(y, x, v) + \frac{v}{t}\right]\Delta$$

$$+ \varphi'_y(y, x, v) a(y, x)\Delta + o(\Delta)\,.$$

Similarly, we get

$$\varphi'_y(\bar{y}, x, v + \Delta v^{\varepsilon}) = \varphi'_y(y, x, v)$$

$$+ \varphi''_{yv}(y, x, v)\left[\varepsilon^{-1}\frac{\alpha}{\sqrt{t}} G(y, x, v) + \frac{v}{t}\right]\Delta$$

$$+ \varphi''_{yy}(y, x, v) a(y, x)\Delta + o(\Delta)$$

and

$$\varphi''_{yv}(\bar{y}, x, v + \Delta v^{\varepsilon}) = \varphi''_{yy}(y, x, v)$$

$$+ \varphi'''_{yyv}(y, x, v)\left[\varepsilon^{-1}\frac{\alpha}{\sqrt{t}} G\left(y, x, \frac{\varepsilon}{\sqrt{t}} u\right) + \frac{v}{t}\right]\Delta$$

$$+ \varphi^{IV}_{yyyy}(y, x, v) a(y, x)\Delta + o(\Delta)\,.$$

This yields

$$\varphi(y + \Delta y^{\varepsilon}, x, v + \Delta v^{\varepsilon}) = \varphi(y, x, v)$$

$$+ \varphi'_v(y, x, v)\left[\varepsilon^{-1}\frac{\alpha}{\sqrt{t}} G\left(y, x, \frac{\varepsilon}{\sqrt{t}} v\right) + \frac{v}{2t}\right]\Delta + \varphi'_y(y, x, v)\, a(y, x)\Delta$$

$$+ \left[\varphi'_y(y, x, v) + \varphi''_{yv}(y, x, v)\left[\varepsilon^{-1}\frac{\alpha}{\sqrt{t}} G\left(y, x, \frac{\varepsilon}{\sqrt{t}} v\right) + \frac{v}{t}\right]\Delta\right.$$

$$\left.+ \varphi''_{yy}(y, x, v) a(y, x)\Delta\right]\int_t^{t+\Delta} \sigma\left(y^{\varepsilon}(s), x_s^{\varepsilon}, \frac{\varepsilon}{\sqrt{s}} v^{\varepsilon}(s)\right) dw(s)$$

$$+\ \frac{1}{2}\left[\ \varphi''_{yy}(y,x,v)+\varphi'''_{yyv}(y,x,v)\left[\ \varepsilon^{-1}\frac{\alpha}{\sqrt{t}}\,G\left(\ y,x,\frac{\varepsilon}{\sqrt{t}}\,v\ \right)+\frac{v}{2t}\ \right]\Delta\right.$$

$$\left.+\ \varphi^{IV}_{yyyy}(y,x,v)a(y,x)\Delta\ \right]\left[\ \int_t^{t+\Delta}\sigma\left(\ y^{\varepsilon}(s),x_s^{\varepsilon},\frac{\varepsilon}{\sqrt{s}}\,v^{\varepsilon}(s)\ \right)dw(s)\ \right]^2$$

$$+\ o(\Delta)\,.$$

This enables us to deduce the following relation for the first term in (10):

$$E_{y,x,v}[\varphi(y+\Delta y^{\varepsilon},x,v+\Delta v^{\varepsilon})-\varphi(y,x,v)](1-\varepsilon^{-2}q(x)\Delta+o(\Delta))$$

$$=\varphi'_v(y,x,v)\left[\ \varepsilon^{-1}\frac{\alpha}{\sqrt{t}}\,G\left(\ y,x,\frac{\varepsilon}{\sqrt{t}}\,v\ \right)+\frac{v}{t}\ \right]\Delta$$

$$+\ \varphi'_y(y,x,v)a(y,x)\Delta+\frac{1}{2}\,\varphi''_{yy}(y,x,v)\sigma^2\left(\ y,x,\frac{\varepsilon}{\sqrt{t}}\,v\ \right)\Delta+o(\Delta)\,.$$

Similarly, we obtain the following relation for the second term in (10):

$$E_{y,x,v}\left[\ \varphi\left(\ y+\Delta y^{\varepsilon},x_{t+\Delta}^{\varepsilon},v+\Delta v^{\varepsilon}\ \right)-\varphi(y,x,v)\ \right]\left[\ \varepsilon^{-2}q(x)\Delta+o(\Delta)\ \right]$$

$$=\ \ \varepsilon^{-2}q(x)E_{y,x,v}\left[\ \varphi(y,x_{t+\Delta}^{\varepsilon},v)-\varphi(y,x,v)\ \right]+o(\Delta)\,.$$

Thus, according to the definition, for the generator of process (8), we get

$$\boldsymbol{L}_t^{\varepsilon}(x)\varphi(y,x,v):=\lim_{\Delta\to 0}\frac{1}{\Delta}\,E_{y,x,v}\left[\ \varphi\left(\ y+\Delta y^{\varepsilon},x_{t+\Delta}^{\varepsilon},v+\Delta v^{\varepsilon}\ \right)-\varphi(y,x,v)\ \right]$$

$$=\ \ \varepsilon^{-2}\boldsymbol{Q}\varphi(y,x,v)+\boldsymbol{V}_t^{\varepsilon}(x)\varphi(y,x,v)\,,$$

where

$$\boldsymbol{V}_t^{\varepsilon}(x)\,\varphi(y,x,v)=\varphi'_v(y,x,v)\left[\ \varepsilon^{-1}\frac{\alpha}{\sqrt{t}}\,G\left(\ y,x,\frac{\varepsilon}{\sqrt{t}}\,v\ \right)+\frac{\nu}{2t}\ \right]$$

$$+\ \varphi'_y(y,x,v)a(y,x)+\frac{1}{2}\,\varphi''_{yy}(y,x,v)\sigma^2\left(\ y,x,\frac{\varepsilon}{\sqrt{t}}\,v\ \right).$$

**Lemma 3.** *On the test functions* $\varphi(y,x,v)\in C^{3,0,3}(R,X,R)$, *the generator* $\boldsymbol{L}_t^{\varepsilon}(x)$ *admits the following asymptotic representation:*

$$\boldsymbol{L}_t^{\varepsilon}(x)\varphi(y,x,v)=\varepsilon^{-2}\boldsymbol{Q}\varphi(y,x,v)+\varepsilon^{-1}\frac{1}{\sqrt{t}}\boldsymbol{G}_0(y,x)\varphi(y,x,v)$$

$$+\frac{1}{t}\boldsymbol{V}(y,x)\varphi(y,x,v)+\boldsymbol{A}(y,x)\varphi(y,x,v)+o(\varepsilon)\,, \qquad (11)$$

*where*

$$\boldsymbol{G}_0(y,x)\varphi(y,x,v) = \alpha G(y,x,0)\varphi'_v(y,x,v)\,,$$

$$\boldsymbol{V}(y,x)\varphi(y,x,v) = v\left(\alpha G'_v(y,x,0)+\frac{1}{2}\right)\varphi'_v(y,x,v)\,,$$

$$\boldsymbol{A}(y,x)\varphi(y,x,v) = a(y,x)\varphi'_y(y,x,v)+\frac{1}{2}\sigma^2(y,x,0)\varphi''_{yy}(y,x,v)\,.$$

***Proof.*** By using the decompositions

$$G\left(y,x,\frac{\varepsilon}{\sqrt{t}}v\right) = G(y,x,0)+G'_v(y,x,0)\frac{\varepsilon}{\sqrt{t}}v+o(\varepsilon)\,,$$

$$\sigma^2\left(y,x,\frac{\varepsilon}{\sqrt{t}}v\right) = \sigma^2(y,x,0)+o(\varepsilon)\,,$$

we derive relation (11) from (9).

Consider a perturbed test function

$$\varphi_t^{\varepsilon}(y,x,v) = \varphi(y,v)+\varepsilon\frac{1}{\sqrt{t}}\varphi_1(y,x,v)+\varepsilon^2\frac{1}{t}\varphi(y,x,v)\,.$$

**Lemma 4.** *The solution of the problem of singular perturbations for the truncated generator*

$$\boldsymbol{L}_{t_0}^{\varepsilon}(x)\varphi(y,x,v)=\varepsilon^{-2}\boldsymbol{Q}\varphi(y,x,v)+\varepsilon^{-1}\frac{1}{\sqrt{t}}\boldsymbol{G}_0(y,x)\varphi(y,x,v)$$

$$+\frac{1}{t}\boldsymbol{V}(y,x)\varphi(y,x,v)+\boldsymbol{A}(y,x)\varphi(y,x,v) \qquad (12)$$

*on the test functions* $\varphi_t^{\varepsilon}(y,x,v)$ *with* $\varphi(y,x)\in C^{3,3}(R\times R)$ *has the form*

$$\boldsymbol{L}_{t_0}^{\varepsilon}(x)\varphi_t^{\varepsilon}(y,x,v) = \frac{1}{t}\boldsymbol{L}\varphi(y,v)+\varepsilon\theta_t^{\varepsilon}(x)\varphi(y,v)\,, \qquad (13)$$

*where the limit operator* $\boldsymbol{L}$ *is given by the formula*

$$\boldsymbol{L}\varphi(y,v) = v\left(\alpha g(y) + \frac{1}{2}\right)\varphi'_v(y,v) + \frac{1}{2}\alpha^2\sigma_v^2(y)\varphi''_{vv}(y,v)$$

$$+\ t\hat{a}(y)\varphi'_y(y,v) + \frac{t}{2}\hat{\sigma}_y^2(y)\varphi''_{yy}(y,v)\,, \tag{14}$$

$$\hat{a}(y) = \int_X \pi(dx)a(y,x)\,, \qquad \hat{\sigma}_y^2(y) = \int_X \pi(dx)\sigma^2(y,x,0)\,.$$

***Proof.*** According to the scheme of solution of the problem of singular perturbations [6], we compute the value of generator (12) on the perturbed function $\varphi_t^\varepsilon(y,x,v)$ as follows:

$$\boldsymbol{L}_{t_0}^\varepsilon(x)\varphi_t^\varepsilon(y,x,v) = \varepsilon^{-2}\boldsymbol{Q}\varphi(y,v) + \varepsilon^{-1}\frac{1}{\sqrt{t}}[\boldsymbol{Q}\varphi_1(y,x,v)$$

$$+\ \boldsymbol{G}_0(y,x)\varphi(y,v)] + \frac{1}{t}\boldsymbol{Q}\varphi_2(y,x,v) + \frac{1}{t}\boldsymbol{G}_0(y,x)\varphi_1(y,x,v)$$

$$+\ \frac{1}{t}(\boldsymbol{V}(y,x) + t\boldsymbol{A}(y,x))\varphi(y,v) + \varepsilon\theta_t^\varepsilon(x)\varphi(y,v)\,,$$

where

$$\theta_t^\varepsilon(x)\varphi(y,v) = \boldsymbol{A}(y,x)\varphi_1(y,x,v) + \frac{1}{t^{3/2}}\boldsymbol{G}_0(y,x)\varphi_2(y,x,v) + \varepsilon\frac{1}{t^2}\boldsymbol{V}(y,x)\varphi_2(y,x,v)$$

$$+\ \varepsilon\boldsymbol{A}(y,x)\ \varphi_2(y,x,v)\,.$$

Since $\boldsymbol{Q}\varphi(y,v) = 0$, the function $\varphi_1(y,x,v)$ satisfies the equation

$$\boldsymbol{Q}\varphi_1(y,x,v) + \boldsymbol{G}_0(y,x)\varphi(y,v) \ =\ 0\,.$$

By using the balance condition [5], we obtain a solution of this equation in the form

$$\varphi_1(y,x,v) \ =\ \boldsymbol{R}_0\boldsymbol{G}_0(y,x)\varphi(y,v) \ =\ \alpha\boldsymbol{R}_0 G(y,x,0)\varphi'_v(y,v)\,.$$

We now consider the equation for the function $\varphi_2(y,x,v)$, namely,

$$\boldsymbol{Q}\varphi_2(y,v,x) + \boldsymbol{G}_0(y,x)\varphi_1(y,v,x) + (\boldsymbol{V}(y,v) + t\boldsymbol{A}(y,v))\varphi \ =\ \boldsymbol{L}\varphi(y,v)\,, \tag{15}$$

where the limit operator $\boldsymbol{L}$ is determined from the condition of solvability of Eq. (14)

$$\boldsymbol{L} = \Pi \boldsymbol{G}_0(y,x)\boldsymbol{R}_0\boldsymbol{G}_0(y,x) + \Pi \boldsymbol{V}(y,x) + t\Pi \boldsymbol{A}(y,x)\,. \tag{16}$$

Computing the right-hand side of (16), we obtain (14).

Equation (15) admits the representation

$$\boldsymbol{Q}\varphi_2 + \boldsymbol{L}(y,x)\varphi(y,x) = \boldsymbol{L}\varphi(y,x)\,,$$

where

$$\boldsymbol{L}(y,x) = \boldsymbol{G}_0(y,x)\boldsymbol{R}_0\boldsymbol{G}_0(y,x) + (\boldsymbol{V}(y,x) + t\boldsymbol{A}(y,x))\,.$$

By using the last representation and relation (14), we get

$$\varphi_2(y,x,v) = \boldsymbol{R}_0\tilde{\boldsymbol{L}}(y,x)\varphi(y,v)\,,$$

where $\tilde{\boldsymbol{L}}(y,x) = \boldsymbol{L}(y,x) - \boldsymbol{L}$ [8].

***Proof of the Theorem.*** In view of the smoothness of the components of system (1), (2) and representations of the functions $\varphi_1$ and $\varphi_2$, we conclude that the remainder is bounded, i.e.,

$$\left|\theta_t^{\varepsilon}(x)\varphi(y,v)\right| < M\,, \quad M > 0\,. \tag{17}$$

The convergence of the processes $y^{\varepsilon}(t)$ and $v^{\varepsilon}(t)$ to the processes $\xi(t)$ and $\zeta(t)$ follows from (13) and (17) according to the Korolyuk model theorem [6]. Here, the generator of the process $\xi(t)$ has the form

$$\boldsymbol{L}_y\varphi(y,v) = t\hat{a}(y)\varphi'_y(y,v) + \frac{t}{2}\hat{\sigma}_y^2(y)\varphi''_{yy}(y,v)\,.$$

The generator of the limit process $\zeta(t)$ has the form (14) and is the generator of the Ornstein–Uhlenbeck process.

## CONCLUSIONS

We consider the case where the transfer process regarded as a random evolution varies under the influence of Markov switchings along a trajectory of new evolution together with the control. Under the assumption of existence of an equilibrium point of control in the ergodic Markov medium, we construct a procedure of stochastic approximation for this control. For the normalized control, we deduce conditions of asymptotic normality in the form of Ornstein–Uhlenbeck process and establish the generator of the limiting control process.

The obtained result enables one to seek the optimal solution of the problem of control over the diffusion process of transfer with Markov switchings.

## REFERENCES

1. I. I. Gikhman and A. V. Skorokhod, *Controlled Random Processes* [in Russian], Naukova Dumka, Kiev (1977).
2. J. Jacod and A. N. Shiryaev, *Limit Theorems for Stochastic Processes*, Springer, Berlin (1994).
3. M. B. Nevel'son and R. Z. Khas'minskii, S*tochastic Approximation and Recurrence Estimation* [in Russian], Nauka, Moscow (1972).
4. V. B. Kolmanovskii, "Problems of optimal estimation," *Soros. Obrazov. Zh.*, No. 11, 122–127 (1999).
5. A. V. Skorokhod, *Asymptotic Methods of the Theory of Stochastic Differential Equations* [in Russian], Naukova Dumka, Kiev (1987).
6. V. Koroliuk and N. Limnios, *Stochastic Systems in Merging Phase Space*, World Scientific Publishing, Singapore (2005).
7. Ya. M. Chabanyuk, "Asymptotic normality for a continuous procedure of stochastic approximation in Markov media," *Dop. Nats. Akad. Nauk Ukr.*, No 11, 29–34 (2005).
8. Ya. M. Chabanyuk, "Continuous procedure of stochastic approximation in a semi-Markov medium," *Ukr. Mat. Zh.*, **56**, No. 5, 713–720 (2004); ***English translation:*** *Ukr. Math. J.*, **56**, No. 5, 862–872 (2004).